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The symmetry principle in continuum mechanics☆

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Abstract

For problems of the mechanics of an anisotropic inhomogeneous continuum, theorems are given concerning the uninterrupted symmetrical and antisymmetrical analytical continuation of the solution through the plane part of the boundary surface of the medium. Theorems are given for two types of mechanics problem; in the first of these both symmetrical and antisymmetrical continuations of the solution are allowed, while in the second only symmetrical continuation of the solution is allowed. Problems of the first type include problems which are reduced to linear thermoelastic dynamic differential equations of motion of an inhomogeneous anisotropic medium possessing a plane of elastic symmetry, to linear thermoelastic dynamic differential equations of motion of an inhomogeneous Cosserat medium, to non-linear differential equations describing the static elastoplastic stress state of a plate, etc. The second type includes problems which are reduced to non-linear differential equations describing geometrically non-linear strains of shells, to Navier–Stokes equations, etc. These theorems extend the principle of mirror reflection (the Riemann–Schwartz principle of symmetry) to linear and non-linear equations of continuum mechanics. The uninterrupted continuation of the solutions is used to solve problems of the equilibrium state of bodies of complex shape.

The first publication on the uninterrupted continuation of the solution of static boundary problems of elasticity theory through the plane part of the boundary of a body (in the case of a homogeneous isotropic medium), as indicated by Duffin,[1] was the work by Somigliana (1902). In the 1960s, many studies appeared on the analytical continuation of the solution of static boundary problems of elasticity theory for a homogeneous isotropic body through the plane part of the boundary of the body[1,2] and through the spherical part of the boundary.[3–5] For an elastic homogeneous isotropic medium, the boundary conditions of symmetry and antisymmetry and the continuation of solutions through the plane part of the boundary of the body that correspond to them was considered in Refs 1,2.

The new results obtained below include the uninterrupted analytical continuation of the solution of problems of continuum mechanics in the following cases: 1) when the elastic continuum is inhomogeneous and anisotropic, and when this medium is examined in dynamic conditions of strain; 2) when the motion of the continuum is described by non-linear differential equations.

1. Some definitions

Let Ω be a certain region and let Ω^* be its mirror reflection in the plane $y=0$. This means that $M_0^*(x_0, -y_0, z_0) \in \bar{\Omega}^*$ when, and only when, $M_0(x_0, y_0, z_0) \in \bar{\Omega}$ ($\bar{\Omega}$ and $\bar{\Omega}^*$ are the closure of the regions Ω and Ω^*).

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For the variable point $M(x, y, z)$ we will define in the set $\Omega \times \tau$ the function $f(M, t)$, which depends on the position of the point $M(x, y, z)$ in the region Ω and the instant of time $t \in \tau$, $\tau = [t_1, t_2]$ (t_1 and t_2 are constants). If the function $f(M, t)$ is continuously differentiated m times in the set $\Omega \times \tau$, we will write $f(M, t) \in C^m(\Omega \times \tau)$.

The function $\tilde{f}(M^*, t)$ will be termed an even (odd) continuation of the function $f(M, t)$ to the set $\Omega^* \times \tau$ relative to the y coordinate if, at any point $M_0^* \in \Omega^*$ and for any instant of time $t_0 \in \tau$

$$\tilde{f}(M_0^*, t_0) = f(M_0^*, t_0) = (-1)^{k-1} f(M_0, t_0) \tag{1.1}$$

where $k = 1$ ($k = 2$). Here, in the set $(\Omega + \Omega^*) \times \tau$, the function $f(M, t)$ will be called an even (odd) function in y if it is obtained by an even (odd) continuation in y from the set $\Omega \times \tau$ to the set $\Omega^* \times \tau$.

Below, we give without proof [the proofs do not present any fundamental difficulties and are given in: Khomasuridze, N., The effective solution of some three-dimensional boundary and boundary contact problems of thermo-elasticity. Doctorate Dissertation, Tbilisi, 2002.] a number of assertions that are of interest in themselves, are the basis for certain important corollaries in linear and non-linear continuum mechanics and are employed in the formulation of a certain approach, using uninterrupted continuation of the solution, to the solution of problems of the motion and equilibria of bodies of complex shape.

2. Theorem of continuation in the case of classical thermoelasticity

We will consider a system of linear thermoelastic dynamic equations of motion of a body; we will call it system D.

We will introduce the following notation

$$f_0 = f(M_0, t_0), \quad f_0^* = f(M_0^*, t_0) \tag{2.1}$$

Lemma 1. *Let u_0, υ_0, w_0 and θ_0 be the value of the solution of the system D at the point $M_0 \in \Omega$ at the instant of time $t_0 \in \tau$. Then, at the same instant of time t_0, the value of the solution of system D at the mirror-reflected point $M_0^* \in \Omega^*$ will be*

$$u_0^* = (-1)^{k-1} u_0, \quad \upsilon_0^* = -(-1)^{k-1} \upsilon_0, \quad w_0^* = (-1)^{k-1} w_0, \quad \theta_0^* = (-1)^{k-1} \theta_0 \tag{2.2}$$

where $k = 1$ ($k = 2$), if first: 1) 22 elastic and thermal characteristics are continued into region Ω^* evenly, and their derivatives with respect to y are continued oddly relative to the y coordinate [it is assumed that these characteristics belong to the class C^2 (Ω)]; 2) the functions X, Z and $\tilde{T}$ are continued to the set $\Omega^* \times \tau$ evenly (oddly), while the function Y is continued oddly (evenly) relative to y [$X, Y, Z, \tilde{T} \in C^1(\Omega \times \tau)$].

Here, u, υ and w are the x-, y- and z-components of the vector of displacement U, θ is the change in temperature of the body, X, Y and Z are mass forces, $\tilde{T}$ is the intensity of the heat source and $t_0 > 0$.

The elastic body is assumed to be inhomogeneous and anisotropic, and here the anisotropic body possesses a plane of elastic symmetry $y = \text{const}$ and is determined by 13 elastic characteristics, four thermal conductivities, the specific heat under constant strain and four coefficients of linear thermal expansion.[6] Each of these 22 elastic and thermal characteristics is a function of all three coordinates.

Corollary 1. *In the region $\Omega + \Omega^*$ and in the set $(\Omega + \Omega^*) \times \tau$: 1) 22 elastic and thermal characteristics are even functions of y; 2) where $k = 1$ (where $k = 2$), the functions u, w, $X^{(x)}$, $Y^{(y)}$, $Z^{(z)}$, X, Z and $\tilde{T}$ are even (odd), whereas the functions υ, $Z^{(y)}$, $Y^{(z)}$, and Y are odd (even) in y. Here, $X^{(x)}$, $Y^{(y)}$ and $Z^{(z)}$ are the normal stresses, while $X^{(y)} = Y^{(x)}$, $X^{(z)} = Z^{(x)}$ and $Y^{(z)} = Z^{(y)}$ are the shear stresses.*

We will assume that the anisotropic inhomogeneous elastic body occupies a region Ω with a boundary surface containing part S of the plane $y = 0$, and that conditions 1) and 2) of Lemma 1 are satisfied in the region $\Omega + S$ and in the set $(\Omega + S) \times \tau$ respectively. Then, the following theorem concerning symmetrical continuation of the solution at $k = 1$ and antisymmetrical continuation of the solution at $k = 2$ holds.

Theorem 1. *Suppose that, in the plane S, irrespective of time, the conditions of symmetry (antisymmetry)*

$$\upsilon = 0, \quad Z^{(y)} = 0, \quad X^{(y)} = 0, \quad \theta_y = 0 \Leftrightarrow \upsilon = 0, \quad w_y = 0, \quad u_y = 0, \quad \theta_y = 0$$

$$(Y^{(y)} = 0, w = 0, u = 0, \theta = 0 \Leftrightarrow \upsilon_y = 0, w = 0, u = 0, \theta = 0)$$

are satisfied.

Then the functions $u, \upsilon, w, \theta \in C^3[(\Omega + S) \times \tau]$, that is, the solution in the set $(\Omega + S) \times \tau$ of system D and of the equations of system D, differentiated with respect to y, are continued uninterruptedly, together with their first, second and third derivatives, through S to the set $\Omega^* \times \tau$. Here, the solution of system D in the set $\Omega^* \times \tau$ is defined by the equations

$$P_0^* = (-1)^{k-1}P_0, \quad Q_0^* = -(-1)^{k-1}Q_0$$

where $k = 1$ $(k = 2)$; P_0 denotes any function w, u, θ, $X^{(x)}$, $Y^{(y)}$, $Z^{(z)}$, $X^{(z)}$, and Q_0 denotes any function υ, $Z^{(y)}$, $X^{(y)}$, and the notation (2.1) is taken into account.

Here and below, the subscripts x, y, z and t denote partial derivatives with respect to the corresponding coordinates and time.

3. Continuation theorem in the case of a Cosserat medium

Following the moment theory of elasticity,[7] we will now adopt the thermoelastic dynamic state of the inhomogeneous isotropic body, which occurs in Section 2, and characterized in the case under examination by the modulus of elasticity $E(x, y, z)$, Poisson's ratio $\nu(x, y, z)$ and five new elastic characteristics $\sigma^{(j)} = \sigma^{(j)}(x, y, z)$, where $j = 1, 2, 3, 4, 5$ $\{\sigma^{(j)} \in C^2[(\Omega + S) \times \tau]$, and $\sigma_y^{(j)}|_S = 0\}$. We will assume that $N^{(xx)}$, $N^{(yy)}$ and $N^{(zz)}$ are normal stresses, $N^{(xy)}$, $N^{(yx)}$, $N^{(yz)}$, $N^{(zy)}$, $N^{(xz)}$ and $N^{(zx)}$ are shear stresses, $\omega^{(x)}$, $\omega^{(y)}$ and $\omega^{(z)}$ are components of the microrotation vector, $M^{(xx)}$, $M^{(yy)}$ and $M^{(zz)}$ are torsional micromoment stresses, $M^{(xy)}$, $M^{(yx)}$, $M^{(yz)}$, $M^{(zy)}$, $M^{(xz)}$ and $M^{(zx)}$ are bending micromoment stresses and $M_i = M_i(M, t) \in C^1[(\Omega + S) \times \tau]$ $(i = 1, 2, 3)$ are components of the mass micromoment vector.

Theorem 2. *Suppose that, in the plane S, irrespective of time, the conditions of symmetry (antisymmetry)*

$$\upsilon = 0, \quad N^{(yz)} = 0, \quad N^{(yx)} = 0, \quad M^{(yy)} = 0, \quad \omega^{(z)} = 0, \quad \omega^{(x)} = 0, \quad \theta_y = 0 \Leftrightarrow$$
$$\Leftrightarrow \upsilon = 0, \quad w_y = 0, \quad u_y = 0, \quad \omega_y^{(y)} = 0, \quad \omega^{(z)} = 0, \quad \omega^{(x)} = 0, \quad \theta_y = 0$$

$$(N^{(yy)} = 0, \quad w = 0, \quad u = 0, \quad \omega^{(y)} = 0, \quad M^{(yz)} = 0, \quad M^{(yx)} = 0, \quad \theta = 0 \Leftrightarrow$$
$$\Leftrightarrow \upsilon_y = 0, \quad w = 0, \quad u = 0, \quad \omega^{(y)} = 0, \quad \omega_y^{(z)} = 0, \quad \omega_y^{(x)} = 0, \quad \theta = 0)$$

are satisfied.

Then the functions $u, \upsilon, \omega, \omega^{(x)}, \omega^{(y)}, \omega^{(z)}, \theta \in C^3[(\Omega + S) \times \tau]$, that is, the solution in the set $(\Omega + \mathrm{S}) \times \tau$ of the system $\tilde{D}$ ($\tilde{D}$ is a system of linear thermoelastic dynamic equations of motion of a Cosserat body) and of the equations of system $\tilde{D}$ differentiated with respect to y, are continued uninterruptedly, together with their first, second and third derivatives, through S to the set $\Omega^* \times \tau$. Here, in the set $\Omega^* \times \tau$

$$\upsilon_0^* = -(-1)^{k-1}\upsilon_0, \quad w_0^* = (-1)^{k-1}w_0, \quad u_0^* = (-1)^{k-1}u_0$$

$$\omega_0^{(y)*} = (-1)^{k-1}\omega_0^{(y)}, \quad \omega_0^{(z)*} = -(-1)^{k-1}\omega_0^{(z)}, \quad \omega_0^{(x)*} = -(-1)^{k-1}\omega_0^{(x)}$$

$$\theta_0^* = (-1)^{k-1}\theta_0$$

where $k = 1$ $(k = 2)$.

First, the functions E, ν, $\sigma^{(j)}$ and Z, X, M_2, $\tilde{T}$ (and Y, M_3, M_1) should be continued respectively into the region Ω^* and to the set $\Omega^* \times \tau$ evenly in y, whereas the functions Y, M_3 and M_1 (the functions Z, X, M_2 and T) should be continued to the set $\Omega^* \times \tau$ oddly in y.

A similar theorem also occurs in the case of the so-called combined moment theory of elasticity (Cosserat's pseudocontinuum).[7]

Note that the requirement given in Theorems 1 and 2, which is that the equations of system D and $\tilde{D}$, and these equations differentiated with respect to y, have to be valid, besides the conditions of symmetry or antisymmetry, in the plane S, is adopted occasionally below when formulating corresponding theorems in non-linear continuum mechanics (in the theorems examined above, this requirement could be removed[8]).

4. Continuation theorems in the case of the strain of plates and shells and the flow of a viscous incompressible fluid

We will further consider various differential equations and systems of differential equations of continuum mechanics with the corresponding boundary conditions. Lemmas and theorems are then given concerning the uninterrupted continuation of solutions of these equations and systems of equations.

4.1. Elastoplastic stresses in a plate

In the two-dimensional region Ω, the definition of the plane stress state that is obtained in a thin plate in the case of small elastoplastic strains, taking into account the hardening of the material, reduces, when there are no mass forces, to the integration of a differential equation in the stress function φ (Ref. 9)

$$\left[\frac{1}{E^{(\varphi)}}\left(\varphi_{xx}-\frac{1}{2}\varphi_{yy}\right)\right]_{xx}+\left[\frac{1}{E^{(\varphi)}}\left(\varphi_{yy}-\frac{1}{2}\varphi_{xx}\right)\right]_{yy}+\left(\frac{1}{E^{(\varphi)}}\varphi_{xy}\right)_{xy} = 0 \tag{4.1}$$

with the corresponding boundary conditions. Here

$$E^{(\varphi)} = E_0\xi^{\frac{a-1}{2a}}, \text{ where } \xi = \varphi_{xx}^2+\varphi_{yy}^2-\varphi_{xx}\varphi_{yy}+3\varphi_{xy}^2$$

and E_0 and a are constants ($E^{(\varphi)}(M)>0$, where $M \in \bar{\Omega}$).

We will denote part of the straight line $y=0$ by S and assume that S is part of the boundary of the region Ω. After this, on S we will examine the conditions of symmetry

$$\upsilon = 0, \quad \sigma^{(xy)} = 0 \tag{4.2}$$

and the conditions of antisymmetry

$$\sigma^{(y)} = 0, \quad u = 0 \tag{4.3}$$

where u and υ are the x- and y-components of the displacement vector, $\sigma^{(x)}$ and $\sigma^{(y)}$ are normal stresses and $\sigma^{(xy)}$ is the shear stress.

It can be proved that the conditions (4.2) and (4.3) are equivalent respectively to the following conditions

$$y = 0: \varphi_y = 0, \quad \varphi_{yyy} = 0 \tag{4.4}$$

$$y = 0: \varphi = 0, \quad \varphi_{yy} = 0 \tag{4.5}$$

4.2. The elastoplastic bending of a plate

In the case of small elastoplastic strains, taking into account the hardening of the material, the stress state of a plate of constant thickness, examined above, is described, under bending, by a differential equation in the deflection w (Ref. 10)

$$\left[D^{(w)}\left(w_{xx}+\frac{1}{2}w_{yy}\right)\right]_{xx}+\left[D^{(w)}\left(w_{yy}+\frac{1}{2}w_{xx}\right)\right]_{yy}+[D^{(w)}w_{xy}]_{xy} = q \quad (4.6)$$

with the corresponding boundary conditions. Here

$$D^{(w)} = D_0\xi^{(a-1)/2}, \quad \xi = w_{xx}^2+w_{yy}^2+w_{xx}w_{yy}+w_{xy}^2$$

D_0 and a are constants ($D^{(w)}(M) > 0$, where $M \in \bar{\Omega}$) and $q = q(x, y)$ is the prescribed load.

The following two types of condition will be examined on S

$$w_y = 0, \quad \tilde{Q}^{(y)} = 0 \text{ and } w = 0, \quad M^{(y)} = 0$$

from which correspondingly the conditions of symmetry

$$y = 0\colon w_y = 0, \quad w_{yyy} = 0 \quad (4.7)$$

and the conditions of antisymmetry

$$y = 0\colon w = 0, \quad w_{yy} = 0 \quad (4.8)$$

follow, where $\tilde{Q}^{(y)}$ is the generalized shearing force and $M(y)$ is the bending moment.

4.3. Non-linear strains of flat orthotropic shells

In the two-dimensional region Ω, the geometrically non-linear stress–strain state of a flat orthotropic shell of positive Gaussian curvature is described by the following equations in the stress function φ and the deflection of the shell w (Ref. 11)

$$\begin{aligned}&\frac{1}{E_2}\varphi_{xxxx}+\frac{1}{E_3}\varphi_{xxyy}+\frac{1}{E_1}\varphi_{yyyy} = -\frac{1}{R_1}w_{yy}-\frac{1}{R_2}w_{xx}-\frac{1}{2}L(w,w)\\&D_1w_{xxxx}+D_3w_{xxyy}+D_2w_{yyyy} = q(x,y)+\frac{h}{R_1}\varphi_{yy}+\frac{h}{R_2}\varphi_{xx}+hL(w,\varphi)\end{aligned} \quad (4.9)$$

Here

$$L(w,\varphi) = w_{yy}\varphi_{xx}-2w_{xy}\varphi_{xy}+w_{xx}\varphi_{yy}, \quad \frac{1}{E_3} = \frac{1}{G}-\frac{2\nu_1}{E_1}$$

$$D_1 = \frac{E_1h^3}{12(1-\nu_1\nu_2)}, \quad D_2 = \frac{E_2h^3}{12(1-\nu_1\nu_2)}, \quad D_3 = 2D_1\nu_2+2D_{12} = 2D_2\nu_1+2D_{12},$$

$$D_{12} = \frac{Gh^3}{6}$$

E_1 and E_2 are the moduli of elasticity, R_1 and R_2 are the principal radii of curvature, h is the shell thickness, ν_1 and ν_2 are Poisson's ratios, G is the shear modulus and $q(x, y)$ is the prescribed load.

Let S be part of the boundary of the region Ω and at the same time part of the straight line $y = 0$. Then the conditions of symmetry on S will have the form

$$y = 0\colon w_y = 0, \quad \tilde{Q}^{(y)} = 0, \quad \upsilon = 0, \quad \sigma_0^{(xy)} = 0 \quad (4.10)$$

where $\tilde{Q}^{(y)}$ is the generalized shearing force, υ is the displacement along the y coordinate and $\sigma_0^{(xy)}$ is the shear force in the middle plane.

It can be shown that conditions (4.10) are equivalent to the conditions

$$y = 0: w_y = 0, \quad w_{yyy} = 0, \quad \varphi_y = 0, \quad \varphi_{yyy} = 0 \tag{4.11}$$

In the isotropic case, when

$$E_1 = E_2 = 2E_3 = E \text{ and } D_1 = D_2 = \frac{1}{2}D_3 = \frac{1}{1-\nu}D_{12} = D$$

Eq. (4.9) are transformed into Marguerre's equations; from Marguerre's equations, when $R_1 \to \infty$ and $R_2 \to \infty$, the well-known Karman equations are obtained.

4.4. The flow of a viscous incompressible fluid

The isothermal flow of a viscous incompressible fluid in the three-dimensional region Ω is described by a Navier–Stokes system of equations

$$\mathbf{V}_t + (\mathbf{V} \cdot \nabla)\mathbf{V} = \mathbf{G} - \frac{1}{\rho}\nabla p + \eta \nabla^2 \mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0 \tag{4.12}$$

where $\mathbf{V} = (n, \upsilon, w)$ is the velocity vector of the fluid in the Cartesian system of coordinate x, y, z; p is the pressure; $t \in [t_1, t_2]$ is the time; ρ is the density; η is the kinematic coefficient of viscosity; and $\mathbf{G} = (X, Y, Z)$ is the mass force density vector.

Let S be part of the plane $y=0$ and at the same time part of the boundary of the region Ω. Then, the conditions of symmetry on S, irrespective of time, will be written in the form

$$y = 0: \upsilon = 0, \quad p^{(yz)} = 0, \quad p^{(yz)} = 0, \quad p_y = 0$$

or

$$y = 0: \upsilon = 0, \quad w_y = 0, \quad u_y = 0, \quad p_y = 0 \tag{4.13}$$

where $p^{(yz)}$ and $p^{(yx)}$ are shear stresses.

4.5. Continuation theorems

Let Ω be a two-dimensional region and Ω^* be its mirror reflection in the straight line $y=0$, i.e. $M_0^*(x_0, -y_0) \in \bar{\Omega}^*$ when, and only when, $M_0(x_0, y_0) \in \bar{\Omega}$, and for now let $y_0 > 0$. Then, for the solutions of Eqs. (4.1) and (4.6), the following lemma holds (we will formulate it for Eq. (4.6)).

We will introduce the notation

$$g_0 = g_0(M_0), \quad g_0^* = g_0(M_0^*) \tag{4.14}$$

Lemma 2. *Let w_0 be the value of the solution of Eq. (4.6) at the point $M_0 \in \Omega$. Then, the value of the solution of Eq. (4.6) at the mirror-reflected point $M_0^* \in \Omega^*$ will be*

$$w_0^* = (-1)^{k-1} w_0 \tag{4.15}$$

where $k=1$ ($k=2$) if the function q is continued into the region Ω^* evenly (oddly) in y.

The lemma concerning the solution of Eq. (4.1) is obtained from Lemma 2 if w in the latter is replaced by φ and if it is assumed that $q=0$.

Theorem 3. *Let S be part of the straight line $y=0$ and at the same time part of the boundary of the region Ω, and let the conditions of symmetry (4.7) (the conditions of antisymmetry (4.8)) be satisfied on S. Then the function*

$w \in C^4(\Omega)$ (the function $w \in C^4(\Omega + S)$), that is, the solution of Eq. (4.6) in the region Ω (in the region $\Omega + S$), is continued uninterruptedly, together with its derivatives with respect to the fourth order, through S into the region Ω^* if the function q is continued evenly (oddly) in y in Ω^*. In this case, in the region Ω^*

$$w_0^* = (-1)^{k-1} w_0, \quad M_0^{(x)*} = (-1)^{k-1} M_0^{(x)}, \quad M_0^{(y)*} = (-1)^{k-1} M_0^{(y)},$$
$$M_0^{(xy)*} = -(-1)^{k-1} M_0^{(xy)}$$

where $k = 1$ ($k = 2$).

Remark 1. The properties of the plate material in Lemma 2 and Theorem 3 are assumed to be identical at the points M_0 and M_0^*, but the function $D^{(w)}$ characterizes the material from which the plate is made, and therefore the conditions

$$y = 0: D_y^{(w)} = 0, \quad D_{yyy}^{(w)} = 0 \tag{4.16}$$

must be satisfied in the case of both symmetrical and antisymmetrical continuation of the solution of Eq. (4.6).

Remark 2. Theorems similar to Theorem 3 can be considered to be valid for the solution of problems (4.1), (4.4) and (4.1), (4.5). Furthermore, since E_φ is a characteristic of the plate material in the case of a plane stress state, it follows that, like conditions (4.16), the following conditions hold

$$y = 0: E_y^{(\varphi)} = 0, \quad E_{yyy}^{(\varphi)} = 0 \tag{4.17}$$

Lemmas and theorems will now be given concerning the continuation of the solution of the systems of differential Eqs. (4.9) and (4.12).

Lemma 3. *Let φ_0 and w_0 be the value of the solution of the system (4.9) at the point $M_0 \in \Omega$. Then, the value of the solution of this system at the mirror-reflected point $M_0^* \in \Omega^*$ will be*

$$\varphi_0^* = \varphi_0, \quad w_0^* = w_0$$

if the prescribed load q is continued into the region Ω^* evenly with respect to the y coordinate.

Theorem 4. *Let S be a part of the straight line $y = 0$ and at the same time part of the boundary of the region Ω, and let the symmetry conditions (4.11) be satisfied on S. Then, the functions φ, $w \in C^4(\Omega)$, i.e. the solution of system of Eq. (4.9) in the region Ω, are continued uninterruptedly, together with their derivatives with respect to the fourth order inclusive, through S into the region Ω^* if the function $q(x, y)$ is continued into the region Ω^* evenly in y.*

Below we will return to the notation (2.1).

Lemma 4. *Let u_0, υ_0, w_0 and p_0 be the value of the solution of system (4.12) at the point $M_0 \in \Omega$ at the instant of time $t_0 \in \tau$. Then, at the same instant of time t_0, the value of the solution of this system at the mirror-reflected point $M_0^* \in \Omega^*$ will be*

$$u_0^* = u_0, \quad \upsilon_0^* = -\upsilon_0, \quad w_0^* = w_0, \quad p_0^* = p_0$$

if the functions X and Z are continued to the set $\Omega^* \times \tau$ evenly, and the function Y is continued oddly in y.

Theorem 5. *Let S be part of the plane $y = 0$ and at the same time part of the boundary of the region Ω, and suppose that on S, irrespective of time, the symmetry conditions (4.13) are satesfied. Then the functions $u, \upsilon, w, p \in C^2(\Omega \times \tau)$, that is, the solution of system of Eq. (4.12) in the set $\Omega \times \tau$, are continued uninterruptedly, together with their first and second derivatives, through S to the set $\Omega^* \times \tau$ if X and Z are continued to the set $\Omega^* \times \tau$ evenly and Y is continued to the set $\Omega^* \times \tau$ oddly in y. Here, the solution of Eq. (4.12) in the set $\Omega^* \times \tau$ is defined by the equations*

$$P_0^* = P_0, \quad Q_0^* = -Q_0$$

where P_0 denotes any function u, w, p, $p^{(xx)}$, $p^{(yy)}$ and $p^{(xz)}$, and Q_0 is any function υ, $p^{(xy)}$ and $p^{(zy)}$.

5. Some conclusions and generalizations

Let us consider a number of corollaries from the boundary conditions of symmetry and antisymmetry that have been widely used in the present paper.

Proposition 1. *If the boundary conditions of symmetry are satisfied in the plane S, it remains a plane after strain, whatever this strain (linear, non-linear, elastic or plastic), since the points of the deformed medium that lie in the plane S before strain remain in S after the strain. In other words, for the doubled region* $\Omega + S + \Omega^*$*, the plane S remains after strain, the same plane S and, moreover, it remains a plane of symmetry for components of the displacement vector and the strain tensor (stress tensor) whatever the strain.*

Consequently, if the boundary conditions of symmetry for the continuum are satisfied, the fundamental principle of symmetry should be satisfied.

Proposition 2. I*f, however, the boundary conditions of antisymmetry are satisfied in the plane S, then this plane, after strain, changes its position and is transformed into a certain non-planar surface; in other words, the points of the deformed medium that lay in the plane S before the strain do not remain in it after the strain. Therefore, the principle of antisymmetry is conditional and occurs only for small displacements and strains.*

We will illustrate this for the case of the theorems set out in the previous section.

Proposition 3. Theorems 1 and 2 *concern linear problems of elasticity for small displacements, strains and angles of rotation, and therefore relate both to symmetrical and antisymmetrical continuation of the solution.*

Theorem 3, which concerns the problem of small elastoplastic bending of plates, although such bending leads to a non-linear differential equation, relates both to symmetrical and antisymmetrical continuation of the solution on account of the smallness of the displacements and strains. In Theorems 4 and 5 the strains (Theorem 4) or displacements (Theorem 5) are not small (finite), and therefore only symmetrical continuation of the solution is given in them, since the corresponding differential equations do not allow antisymmetrical continuation of the solution.

Remark 3. Satisfaction of the principle of symmetrical continuation of the solution with satisfaction of the symmetry conditions on the plane part of the boundary of the medium is obligatory in continuum mechanics; consequently, this principle is one of the important criteria in assessing a particular mathematical model in continuum mechanics problems.

Note that conditions (4.16) and (4.17) given in Remarks 1 and 2 can also be used to evaluate certain mathematical models, in particular for evaluating mathematical models in plasticity theory.

6. A practical application of the theorems concerning the uninterrupted continuation of solutions

We will now touch upon some questions of the practical application of the principles of symmetry and antisymmetry.
Consider a rectangular parallelepiped

$$\Omega = \{0 < x < x_1, 0 < y < y_1, 0 < z < z_1\}$$

of orthotropic material inhomogeneous with respect to z. For simplicity, we will assume that there are no internal and external disturbing fields and mass forces. Then, on the side faces $x = 0$, $x = x_1$, $y = 0$ and $y = y_1$ let the conditions of symmetry be satisfied, on the boundary $z = 0$ let a uniformly distributed normal load q_0 be specified, and in the middle of the face $z = z_1$ on a small area s let a normal load q_1 ($x_1 y_1 q_0 = s q_1$) be uniformly distributed, and on both faces, i.e. at $z = 0$ and $z = z_1$, let the shear stresses be zero.

Suppose there is a solution of the problem of the equilibrium state of the parallelepiped. Then, the solution of problems for bodies with a more complex geometric shape can be reduced to this solution, since a parallelepiped with a definite stress state can serve as the basis (the "building block") for constructing the stress state of a body of complex

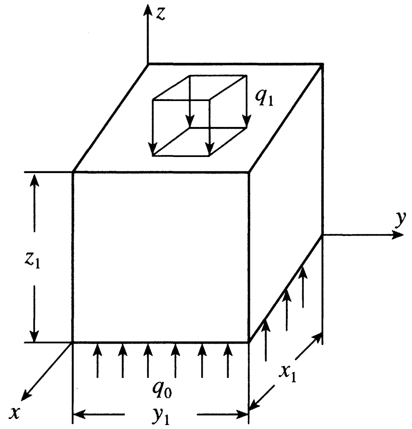

Fig. 1.

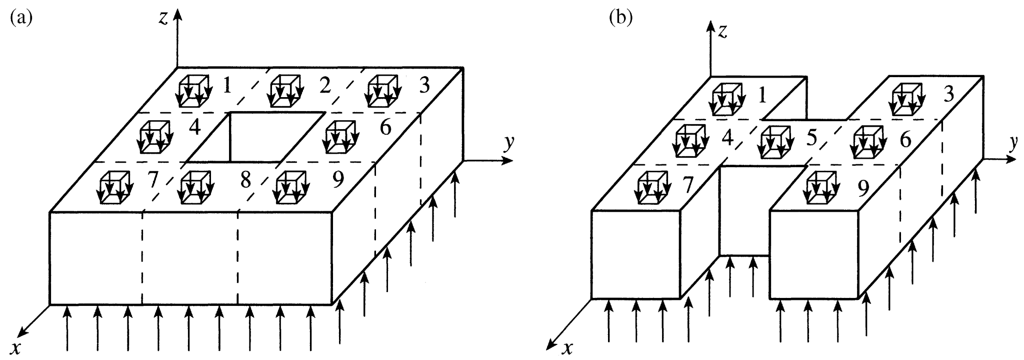

Fig. 2.

shape (true, with a constraint imposed on the surface load). This is so indeed, in view of the fact that the symmetry conditions on the side faces of the parallelepiped make it possible uninterruptedly to continue the solution and to obtain a region of more complex shape.

We will clarify this by means of illustrations.

Fig. 1 shows a rectangular parallelepiped with prescribed loads on its faces $z = 0$ and $z = z_1$. Fig. 2 gives examples of bodies of more complex shape that were obtained as a result of using the parallelepiped shown in Fig. 1 as a "building block". Of course, the stress state of the bodies shown in Fig. 2 are determined by the stress state of the parallelepiped shown in Fig. 1.

It must be pointed out, in particular, that the loads q_0 and q_1 may cause both linear and non-linear strains and both elastic and elastoplastic strains in the parallelepiped.

If the antisymmetry conditions are satisfied on the side faces of the parallelepiped, the load $q_0 = 0$ and the load q_1 causes small displacements and strains, then in Fig. 2 the load q_1 on certain blocks should be in the opposite direction. In particular, in Fig. 2a these are blocks 2, 4, 6 and 8, and in Fig. 2b they are blocks 4 and 6.

The stress state for a parallelepiped/block in some cases can be found effectively (for example, this is done in Ref. 12 for a transversely isotropic parallelepiped), and thereby we will have effective solutions for bodies of complex shape. Such solutions can be used, at the very least, as model solutions for checking the accuracy of a particular approximate method.

As the "building blocks" for constructing the stress state of a body of complex shape, besides a rectangular parallelepiped, it is possible to use other bodies bounded by the coordinate surfaces of the selected system of coordinates.

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